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Note

The transitivity of Conway's M_{13}

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Abstract

J.H. Conway introduced a set of permutations called M_{13} by a game related to the projective plane of order 3. The set M_{13} consists of certain permutations on 13 letters, and contains the Mathieu group M_{12}. W.J. Martin and B.E. Sagan generalized the concept of transitivity for a set of permutations by defining λ-transitivity for each partition λ of the degree of the permutations. We determine the partitions λ of 13 for which the set M_{13} is λ-transitive.

Keywords: Mathieu group; Multiply transitivity; Permutation

1. Introduction

A finite permutation group G is called k-transitive if any k-tuple of distinct points can be mapped, by some element of G, to any other k-tuple of distinct points. Other than the symmetric group S_n and the alternating group A_n which are n-transitive and $(n-2)$-transitive, respectively, the Mathieu groups M_{11}, M_{12}, M_{23} and M_{24} are the only 4-transitive permutation groups, and M_{12}, M_{24} are the only 5-transitive permutation groups. The proof of this fact involves a case-by-case analysis based on the classification of finite simple groups [3, Chapter 7].

Conway [1] introduced a set M_{13} of permutations on 13 letters, which contains the Mathieu group M_{12}, and he claims that M_{13} is 6-transitive in some sense. Martin and Sagan [4] generalized the concept of transitivity for a set of permutations. For a partition λ of a positive integer n, we say that a subset D of the symmetric group S_n is λ-transitive if there exists $r > 0$ such that for any partitions P, Q of shape λ, $\sharp\{\tau \in D \mid P^\tau = Q\} = r$. In particular, a permutation group of degree n is t-transitive if and only if it is $(n-t, 1, \ldots, 1)$-transitive. Since Conway's M_{13} is not $(7, 1, 1, 1, 1, 1, 1)$-transitive on this definition, Martin and Sagan raised the question to determine the full transitivity of M_{13}. We introduce an expression for the elements of M_{13} to answer completely the question of Martin and Sagan.

2. Construction of M_{13}

The set of permutations M_{13} was introduced by Conway [1], and it has recently been further investigated in [2]. Although there is recently introduced notation for the elements of M_{13} in [2], we define our own notation.

Let $\Omega := \{0, 1, \ldots, 11, \infty\}$ be the set of points of a projective plane P of order 3, and $\Omega_{12} := \Omega - \{\infty\}$. For $a \in \Omega_{12}$ let b and c be the remaining points on the projective line through ∞ and a. Denote the double transposition

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$(\infty\, a)(b\, c)$ by $\sigma(a)$. Set $\sigma(\emptyset)=\sigma(\infty)=id_\Omega$. Recursively, for an integer k such that $k\geqslant 2$ and $a_1, a_2, \ldots, a_k \in \Omega$ such that $a_1 \neq a_2 \neq \cdots \neq a_k$, define

$$\sigma(a_1, a_2, \ldots, a_k) := \tau(\infty\, a_k^\tau)(b^\tau\, c^\tau),$$

where $\tau=\sigma(a_1,\ldots,a_{k-1})$ and $\{a_{k-1}, a_k, b, c\}$ is the line determined by a_{k-1}, a_k. We can see that $\sigma(a)$ is the move $a|bc$, and a *triangular permutation* in the sense of [1] is of the form $\sigma(a, b, \infty)$. Furthermore the move sequence $[0, p_1, \ldots, p_n]$ defined in [2] is related to our $\sigma(p_1,\ldots,p_n)$ by the formula

$$[0, p_1, \ldots, p_n] = \sigma(p_1, \ldots, p_n)^{-1}.$$

Note that in this paper we use the convention that a permutation on Ω acts from the right, while the authors of [2] use the opposite convention.

The sets M_{13}, M_{12} are defined as

$$M_{13} := \{\sigma(a_1, \ldots, a_k) \mid k \in \mathbb{N},\ a_i \in \Omega,\ a_i \neq a_{i+1}\ (1\leqslant i\leqslant k-1)\},$$
$$M_{12} := \{\tau \in M_{13} \mid \infty^\tau = \infty\}.$$

The next proposition is useful to describe the elements of M_{13}.

Proposition 1. *Let $a_1,\ldots,a_k, b_1,\ldots,b_l \in \Omega$ be such that $a_1 \neq \cdots \neq a_k \neq \infty \neq b_1 \neq \cdots \neq b_l$. Then*

$$\sigma(a_1, \ldots, a_k, \infty, b_1, \ldots, b_l) = \sigma(b_1, \ldots, b_l)\cdot \sigma(a_1, \ldots, a_k, \infty).$$

Proof. We prove by induction on l. Let

$$\rho = \sigma(b_1, \ldots, b_{l-1}),$$
$$\pi = \sigma(a_1, \ldots, a_k, \infty),$$

so $\sigma(a_1,\ldots,a_k,\infty,b_1,\ldots,b_{l-1}) = \rho\pi$ by the inductive hypothesis. Suppose that the line determined by b_{l-1}, b_l is $\{b_{l-1}, b_l, c, d\}$. Then

$$\begin{aligned}
\sigma(a_1, \ldots, a_k, \infty, b_1, \ldots, b_l) &= \rho\pi(\infty\ b_l^{\rho\pi})(c^{\rho\pi}\ d^{\rho\pi})\\
&= \sigma(b_1, \ldots, b_l)(\infty\ b_l^{\rho})(c^{\rho}\ d^{\rho})\pi(\infty\ b_l^{\rho\pi})(c^{\rho\pi}\ d^{\rho\pi})\\
&= \sigma(b_1, \ldots, b_l)\pi(\infty^{\pi}\ b_l^{\rho\pi})(c^{\rho\pi}\ d^{\rho\pi})(\infty\ b_l^{\rho\pi})(c^{\tau}\ d^{\rho\pi})\\
&= \sigma(b_1, \ldots, b_l)\cdot\sigma(a_1, \ldots, a_k, \infty). \qquad \square
\end{aligned}$$

The following propositions are obvious.

Proposition 2. *If i is an integer such that $1\leqslant i\leqslant k$ and $x \in \Omega - \{a_i\}$, then*

$$\sigma(a_1, \ldots, a_k) = \sigma(a_1, \ldots, a_i, x, a_i, a_{i+1}, \ldots, a_k).$$

Proposition 3. *For $a, b \in \Omega_{12}$ such that $\{a, b, \infty\}$ is contained in a line,*

$$\sigma(a, \infty) = \sigma(a, b, \infty) = id_\Omega.$$

With these propositions, we prove the following theorem.

Theorem 4. *M_{12} is the group generated by triangular permutations, and*

$$M_{13} = \coprod_{a\in\Omega} \sigma(a) M_{12}.$$

Proof. Let $\alpha = \sigma(a_1, \ldots, a_k)$. Then $\alpha \in M_{12}$ if and only if $a_k = \infty$. For i in $\{1, \ldots, k-1\}$, if $a_i \neq \infty$ then we insert ∞, a_i between a_i and a_{i+1} by Proposition 2. So by Propositions 1 and 3, α is written as a product of triangular permutations.

If $a_k = \infty$, then $\alpha \in M_{12}$. Otherwise, Proposition 2 implies

$$\alpha = \sigma(a_1, \ldots, a_k, \infty, a_k)$$

so $\alpha \in \sigma(a_k)M_{12}$ by Proposition 1. $\square$

3. Transitivity of M_{13}

In this section, we define transitivity for a set of permutations as in [4], and show that M_{13} is not $(7, 6)$-transitive. This enables us to determine its full transitivity by Theorem 7.

An integer tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called a integer partition of a positive integer n if $\lambda_i \geqslant \lambda_{i+1} > 0$ and $\sum_{i=1}^{k} \lambda_i = n$. A partition $P = (P_1, \ldots, P_k)$ of the set $\Omega_n := \{1, \ldots, n\}$ is said to have shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, if $|P_i| = \lambda_i$.

Definition 5. Let n be an integer and D be a set of permutations on Ω_n. For an integer partition λ of n, we say that D is λ-transitive if there exists $r > 0$ such that for any set partitions P, Q of shape λ, $\sharp\{\tau \in D \mid P^\tau = Q\} = r$.

For example, a permutation group G is t-transitive on Ω_n if and only if G is $(n - t, 1^t)$-transitive, where 1^t means $\underbrace{1, \ldots, 1}_{t}$.

It might seem that the transitivity as defined above is too restrictive, and one may wish to replace the condition by $\sharp\{\tau \in D \mid P^\tau = Q\} > 0$. But later we show that for some partitions P, Q of shape $(7, 6)$, $\sharp\{\tau \in M_{13} \mid P^\tau = Q\} = 0$, and this implies that M_{13} is not $(7, 6)$-transitive even with the relaxed definition.

We first prove the following general result.

Lemma 6. *For each $i \in \Omega_{n+1} = \{1, \ldots, n+1\}$, let a_i be a permutation on Ω_{n+1} such that $i^{a_i} = n+1$, and G be a permutation group on $\Omega_n = \{1, \ldots, n\}$. If G is $(n - t, 1^t)$-transitive on Ω_n and*

$$D = \bigcup_{i \in \Omega_{n+1}} a_i G,$$

then D is a $(n - t + 1, 1^t)$-transitive set on Ω_{n+1}.

Proof. For t-tuples $X = (x_1, \ldots, x_t)$, $Y = (y_1, \ldots, y_t)$ of distinct elements of Ω_{n+1}, we define

$$D_Y^X := \{\tau \in D \mid X^\tau = Y\}.$$

First, we suppose that Y contains $n + 1$, for example, $y_1 = n + 1$. Then $D_Y^X \subset a_{x_1}G$, and $\{x_k^{a_{x_1}} \mid 2 \leqslant k \leqslant t\}, \{y_k \mid 2 \leqslant k \leqslant t\} \subset \Omega_n$. By the $(n - (t-1), 1^{t-1})$-transitivity of G on Ω_n,

$$\begin{aligned} |D_Y^X| &= \sharp\{g \in G \mid (x_k^{a_{x_1}})^g = y_k \ (2 \leqslant k \leqslant t)\} \\ &= \frac{|G|}{n \cdot (n-1) \cdots (n - (t-2))}. \end{aligned}$$

Next, we assume that $n + 1$ does not appear in Y. For an integer i such that $1 \leqslant i \leqslant t$, if $a_i g \in D_Y^X \cup a_i G$ then $i \notin X$ and $\{x_1^{a_i}, \ldots, x_t^{a_i}\}, Y \subset \Omega_n$, so by the $(n - t, 1^t)$-transitivity of G on Ω_n,

$$\begin{aligned} |D_Y^X| &= \sum_{i \in \Omega_{n+1} - X} \sharp\{g \in G \mid (X^{a_i})^g = Y\} \\ &= |\Omega_{n+1} - X| \cdot \frac{|G|}{n \cdot (n-1) \cdots (n - (t-1))} \\ &= \frac{|G|}{n \cdot (n-1) \cdots (n - (t-2))}. \qquad \square \end{aligned}$$

By this lemma, we see that M_{13} is $(8, 1^5)$-transitive. We will show that M_{13} is not $(7, 6)$-transitive.

If M_{13} is $(7, 6)$-transitive, then for any 6-element sets P, Q of Ω_{13},

$$\sharp\{\tau \in M_{13} \mid P^{\tau} = Q\} = \frac{|M_{13}|}{\binom{13}{6}} = 720.$$

It is known that M_{12} leaves the set of hexads invariant (see [1] for details). We define $H := \{h^{\sigma(a)} \mid a \in \Omega, h : \text{hexad}\}$. If $h = \{1, 2, 3, 4, 5, 6\}$ then $h^{\sigma(7)} = h^{\sigma(8)}$ and

$$|H| < |\Omega| \cdot 132 = \binom{13}{6},$$

so there is a 6-element set P that is not contained in H. Taking Q as a hexad, we obtain

$$\sharp\{\tau \in M_{13} \mid P^{\tau} = Q\} = 0.$$

Therefore M_{13} is not $(7, 6)$-transitive.

We need to introduce the dominance order on integer partitions of n, in order to state a result of Martin and Sagan [4]. For two integer partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_l)$, we define

$$\lambda \trianglelefteq \mu \iff \sum_{i=1}^{j} \lambda_i \leqslant \sum_{i=1}^{j} \mu_i \quad \text{for any positive integer } j$$

where $\lambda_i = 0$ for $i \geqslant k$ and $\mu_i = 0$ for $i \geqslant l$.

Theorem 7 (*Martin and Sagan [4]*). *If a set D of permutations is λ-transitive and $\lambda \trianglelefteq \mu$, then D is μ-transitive.*

Using this theorem, we can determine the transitivity of M_{13}.

Theorem 8. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be an integer partition of* 13. *Then M_{13} is λ-transitive if and only if $\lambda_1 \geqslant 8$.*

Proof. For any integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of 13, if $\lambda_1 \geqslant 8$ then $\lambda \trianglerighteq (8, 1^5)$ because $\lambda_i > 0$ for any i from 1 to k. M_{13} is $(8, 1^5)$-transitive, so M_{13} is also λ-transitive by Theorem 7. And if $\lambda_1 \leqslant 7$ we suppose M_{13} is λ-transitive. Then M_{13} is $(7, 6)$-transitive by Theorem 7 again since $\lambda \trianglelefteq (7, 6)$. But M_{13} is not $(7, 6)$-transitive, therefore M_{13} is not λ-transitive. □

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References

[1] J.H. Conway, M_{13}, Surveys in combinatorics, London Mathematical Society Lecture Note Series, vol. 241, Cambridge University Press, Cambridge, 1997, pp. 1–11.
[2] J.H. Conway, N.D. Elkies, J.L. Martin, The Mathieu group M_{12} and its pseudogroup extension M_{13}, Experiment. Math. 15 (2) (2006) 223–236.
[3] J.D. Dixon, B. Mortimer, Permutation groups, Springer, New York, 1996.
[4] W.J. Martin, B.E. Sagan, A new notion of transitivity for sets of permutations, J. London Math. Soc. 73 (1–2) (2006) 1–13.